Iris: A Modular Foundation for Higher-Order Concurrent Separation Logic

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Language-independent higher-order separation logic with simple foundations for verifying fine-grained concurrent programs in Coq.

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- **Simple foundations:** Small, "canonical" set of primitive rules
- **Coq:** Provides practical support for machine-checked proof

The versatility of Iris

Iris has morphed from a failed book project into a serious development platform for multiple ongoing efforts in program verification:

- The Rust type system (Jung, Jourdan, Krebbers, Dreyer)
- Logical relations (Krogh-Jespersen, Svendsen, Timany, Birkedal, Krebbers)
- Termination-preserving refinement (Tassarotti, Jung, Harper)
- Weak memory concurrency (Kaiser, Dang, Dreyer, Lahav, Vafeiadis)
- Object capability patterns (Swasey, Garg, Dreyer)
- Logical atomicity (Jung, Swasey, Krogh-Jespersen, Zhang, Dreyer, Birkedal)
- Defining Iris (Krebbers, Jung, Jourdan, Bizjak, Dreyer, Birkedal)

Most of these projects are formalized in Iris in  Coq

Prologue: A brief history of Iris

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- We had just developed a logic called CaReSL [ICFP'13] for verifying contextual refinement for fine-grained concurrent data structures, but it was somewhat ad-hoc and didn't feel canonical enough for a pedagogical text.

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- Aaron has a bright idea – Simplify foundations of advanced concurrent separation logics (CSLs) down to 2 mechanisms:
  - **Partial commutative monoids (PCMs)**
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  - **Invariants** for tying ghost state to physical state
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- Around the end of 2013, I gave this idea to my new student, **Ralf Jung**, to play with. He picked it up and ran with it.
  - Developed encodings of sophisticated proof rules from various advanced CSLs in terms of PCMs and invariants
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Iris is born!

Some obvious questions you might have about Iris 1.0

1. Are PCMs-and-invariants the "right" foundation?
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Some obvious questions you might have about Iris 1.0

1. Are PCMs-and-invariants the "right" foundation?
   - Are there things it can't express? **Yes.**
   - Is it as simple and canonical as possible? **No.**

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     Coq support!

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Iris 2.0 [ICFP'16]: Making Iris more **expressive**

- Extends Iris 1.0 with "higher-order ghost state", i.e., ghost state that can store (Iris) propositions
- Generalizes PCMs to "cameras" (step-indexed PCMs)

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- Distills essence of Iris to a bare-bones modal base logic, which knows nothing about programs
- Entire program specification layer of Iris (e.g., Hoare triples) is *encoded* on top of the base logic

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Iris proof mode [POPL'17]: Making Iris more **usable**

- Builds tactical support for not only proving soundness of Iris in Coq, but doing **proofs of programs in Iris in Coq**
- Fundamental catalyst for all the ongoing applications of Iris!

Preview of the rules of the Iris base logic

Laws of (affine) bunched implications

$$\begin{array}{c} \mathsf{True} * P \dashv\vdash P \\ P * Q \vdash Q * P \\ (P * Q) * R \vdash P * (Q * R) \end{array} \qquad \frac{P_1 \vdash Q_1 \qquad P_2 \vdash Q_2}{P_1 * P_2 \vdash Q_1 * Q_2} \qquad \frac{P * Q \vdash R}{P \vdash Q \mathrel{-\!\!*} R} \qquad \frac{P \vdash Q \mathrel{-\!\!*} R}{P * Q \vdash R}$$

Laws for resources and validity

$$\mathsf{Own}(a) * \mathsf{Own}(b) \dashv\vdash \mathsf{Own}(a \cdot b) \qquad \mathsf{True} \vdash \mathsf{Own}(\varepsilon) \qquad \mathsf{Own}(a) \vdash \Box\, \mathsf{Own}(|a|)$$
$$\mathsf{Own}(a) \vdash \mathcal{V}(a) \qquad \mathcal{V}(a \cdot b) \vdash \mathcal{V}(a) \qquad \mathcal{V}(a) \vdash \Box\, \mathcal{V}(a)$$

Laws for the basic update modality

$$\frac{P \vdash Q}{\Rrightarrow P \vdash \Rrightarrow Q} \qquad P \vdash \Rrightarrow P \qquad \Rrightarrow \Rrightarrow P \vdash \Rrightarrow P$$

$$Q * \Rrightarrow P \vdash \Rrightarrow (Q * P) \qquad \frac{a \rightsquigarrow B}{\mathsf{Own}(a) \vdash \Rrightarrow \exists b \in B.\, \mathsf{Own}(b)}$$

Laws for the always modality

$$\frac{P \vdash Q}{\Box P \vdash \Box Q} \qquad \Box P \vdash P \qquad \begin{array}{c} \mathsf{True} \vdash \Box\, \mathsf{True} \\ \Box (P \wedge Q) \vdash \Box (P * Q) \\ \Box P \wedge Q \vdash \Box P * Q \end{array} \qquad \begin{array}{c} \Box P \vdash \Box \Box P \\ \forall x.\ \Box P \vdash \Box \forall x.\ P \\ \Box \exists x.\ P \vdash \exists x.\ \Box P \end{array}$$

Laws for the later modality

$$\frac{P \vdash Q}{\triangleright P \vdash \triangleright Q} \qquad (\triangleright P \Rightarrow P) \vdash P \qquad \begin{array}{c} \forall x.\ \triangleright P \vdash \triangleright \forall x.\ P \\ \triangleright \exists x.\ P \vdash \triangleright \mathsf{False} \vee \exists x.\ \triangleright P \end{array} \qquad \begin{array}{c} \triangleright (P * Q) \dashv\vdash \triangleright P * \triangleright Q \\ \Box \triangleright P \dashv\vdash \triangleright \Box P \end{array}$$

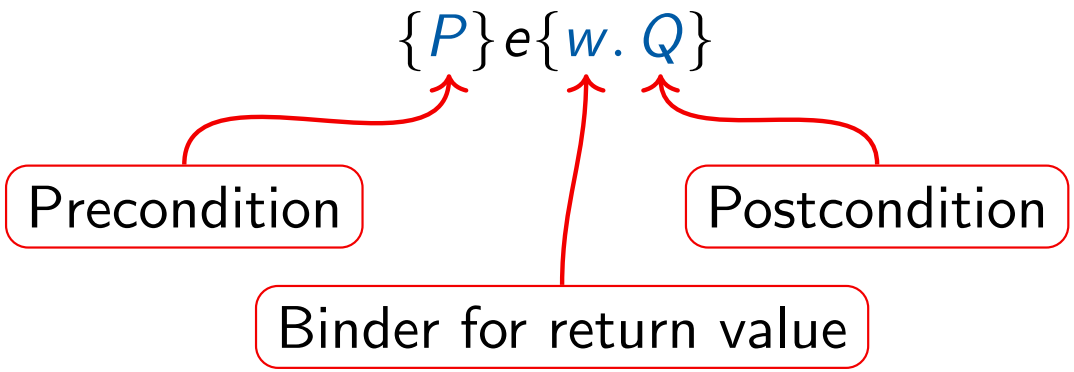
Laws for timeless assertions

$$\triangleright P \vdash \triangleright \mathsf{False} \vee (\triangleright \mathsf{False} \Rightarrow P) \qquad \triangleright \mathsf{Own}(a) \vdash \exists b.\, \mathsf{Own}(b) \wedge \triangleright (a = b)$$

The Iris story, Part 1:
Working with ghost state and invariants

Hoare triples

Hoare triples for partial program correctness:



If the initial state satisfies P, then:

- e does not get stuck/crash
- if e terminates with value v, the final state satisfies $Q[v/w]$

Separation logic [O'Hearn, Reynolds, Yang]

The points-to connective $x \mapsto v$

- provides the knowledge that location x has value v, and
- provides exclusive ownership of x

Separating conjunction $P * Q$: the state consists of *disjoint parts* satisfying P and Q

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Example:

$$\{x \mapsto v_1 * y \mapsto v_2\}\mathit{swap}(x, y)\{w.\, w = () \wedge x \mapsto v_2 * y \mapsto v_1\}$$

the $*$ ensures that x and y are different

Concurrent separation logic [O'Hearn, Brookes]

The *par* rule:

$$\frac{\{P_1\}\, e_1\, \{Q_1\} \qquad \{P_2\}\, e_2\, \{Q_2\}}{\{P_1 * P_2\}\, e_1 \,||\, e_2\, \{Q_1 * Q_2\}}$$

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For example:

$$\{x \mapsto 4 * y \mapsto 6\}$$

$$x := !\, x + 2 \;\Big\|\; y := !\, y + 2$$

$$\{x \mapsto 6 * y \mapsto 8\}$$

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Works great for concurrent programs without shared memory: concurrent quick sort, concurrent merge sort, ...

What about shared state/racy programs?

A classic problem:

$$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$$

$$\texttt{fetchandadd}(x, 2) \;\Big\|\; \texttt{fetchandadd}(x, 2)$$

$$!\, x$$

Where $\texttt{fetchandadd}(x, y)$ is the atomic version of $x := !\, x + y$.

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Problem: can only give ownership of x to one thread

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Invariant allocation:

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The invariant assertion $\boxed{R}^{\mathcal{N}}$ expresses that R is maintained as an invariant on the state

Invariant opening:

$$\frac{\{R * P\}\, e\, \{R * Q\}_{\mathcal{E}} \qquad e \text{ atomic}}{\boxed{R}^{\mathcal{N}} \vdash \{P\}\, e\, \{Q\}_{\mathcal{E} \uplus \mathcal{N}}}$$

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Other technical detail: the later $\triangleright$ is needed to support impredicative invariants, i.e., $\boxed{\ldots \boxed{R}^{\mathcal{N}_2} \ldots}^{\mathcal{N}_1}$

Invariants in action

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Problem: still cannot prove it returns 4

Ghost variables

Consider the invariant:

$$\boxed{\exists n.\, x \mapsto n * \ldots}$$

How to relate the quantified value to the state of the threads?

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Ghost variables are allocated in pairs:

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$$\boxed{\exists n_1, n_2.\, x \mapsto (n_1 + n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$$

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$$\text{True} \quad \Rrightarrow \quad \exists \gamma.\ \underbrace{\gamma \hookrightarrow_\bullet n}_{\text{in the invariant}} \quad * \quad \underbrace{\gamma \hookrightarrow_\circ n}_{\text{in the Hoare triple}}$$

When you own both parts you obtain that the values are equal and can update both parts:

$$\gamma \hookrightarrow_\bullet n * \gamma \hookrightarrow_\circ m \quad \Rightarrow \quad n = m$$

$$\gamma \hookrightarrow_\bullet n * \gamma \hookrightarrow_\circ m \quad \Rrightarrow \quad \gamma \hookrightarrow_\bullet n' * \gamma \hookrightarrow_\circ n'$$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$

$\texttt{fetchandadd}(x, 2)$

$\|$

$\texttt{fetchandadd}(x, 2)$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

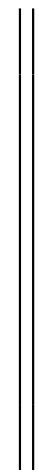
$\texttt{fetchandadd}(x, 2)$ $\quad\|\quad$ $\texttt{fetchandadd}(x, 2)$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$



$\texttt{fetchandadd}(x, 2)$

$\texttt{fetchandadd}(x, 2)$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\texttt{fetchandadd}(x, 2)$ || $\texttt{fetchandadd}(x, 2)$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$

$\texttt{fetchandadd}(x, 2)$

$\{\gamma_2 \hookrightarrow_\circ 0\}$

$\texttt{fetchandadd}(x, 2)$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$
$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$
$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1 + n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$ || $\{\gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1 + n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$

$\texttt{fetchandadd}(x, 2)$ || $\texttt{fetchandadd}(x, 2)$

$\{\gamma_1 \hookrightarrow_\circ 2\}$ || $\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1 + n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
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$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1 + n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$
$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

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$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
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$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$
$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$
$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$

$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$

$\{x \mapsto 0\}$

$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$

allocate $\boxed{\exists n_1, n_2.\, x \mapsto n_1 + n_2 * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2}$

$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$

$!\, x$

$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
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$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\texttt{fetchandadd}(x, 2)$	$\texttt{fetchandadd}(x, 2)$
$\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\{\ldots\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$
$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$
$!\, x$
$\{n.\, n = 4 \wedge \gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$
$\{n.\, n = 4\}$

Ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$
$\{x \mapsto 0\}$
$\{x \mapsto 0 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\circ 0\}$
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$\{\gamma_1 \hookrightarrow_\circ 0 * \gamma_2 \hookrightarrow_\circ 0\}$

$\{\gamma_1 \hookrightarrow_\circ 0\}$	$\{\gamma_2 \hookrightarrow_\circ 0\}$
$\quad\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\quad\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto n_2 * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\quad\{\ldots\}$
$\quad\texttt{fetchandadd}(x, 2)$	$\quad\texttt{fetchandadd}(x, 2)$
$\quad\{\gamma_1 \hookrightarrow_\circ 0 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 0 * \gamma_2 \hookrightarrow_\bullet n_2\}$	$\quad\{\ldots\}$
$\quad\{\gamma_1 \hookrightarrow_\circ 2 * x \mapsto (2+n_2) * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet n_2\}$	
$\{\gamma_1 \hookrightarrow_\circ 2\}$	$\{\gamma_2 \hookrightarrow_\circ 2\}$

$\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2\}$
$\quad\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto (n_1+n_2) * \gamma_1 \hookrightarrow_\bullet n_1 * \gamma_2 \hookrightarrow_\bullet n_2\}$
$\quad\{\gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$
$\quad !\, x$
$\quad\{n.\, n = 4 \wedge \gamma_1 \hookrightarrow_\circ 2 * \gamma_2 \hookrightarrow_\circ 2 * x \mapsto 4 * \gamma_1 \hookrightarrow_\bullet 2 * \gamma_2 \hookrightarrow_\bullet 2\}$
$\{n.\, n = 4\}$

Ghost variables with fractional permissions [Boyland]

What if we have n threads? Using n different ghost variables, results in different proofs for each thread. *That is not modular.*

Better way: ghost variables with a *fractional permission* $(0, 1]_{\mathbb{Q}}$:

$$\gamma \xhookrightarrow{\pi_1+\pi_2}_{\circ} (n_1 + n_2) \quad \Leftrightarrow \quad \gamma \xhookrightarrow{\pi_1}_{\circ} n_1 * \gamma \xhookrightarrow{\pi_2}_{\circ} n_2$$

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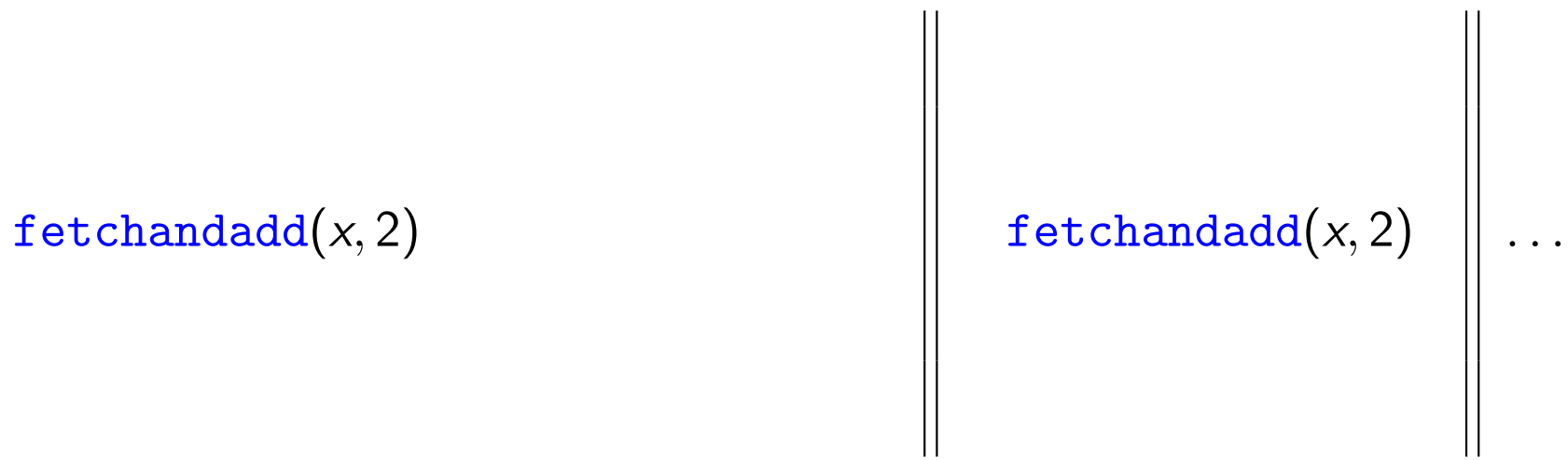
Keeps the invariant that all $\gamma \overset{\pi_i}{\hookrightarrow}_\circ n_i$ sum up to $\gamma \hookrightarrow_\bullet \sum n_i$

Fractional ghost variables in action

$\{\mathsf{True}\}$
$\texttt{let}\, x = \texttt{ref}(0)\, \texttt{in}$



$!\, x$

$\{n.\, n = 2k\}$

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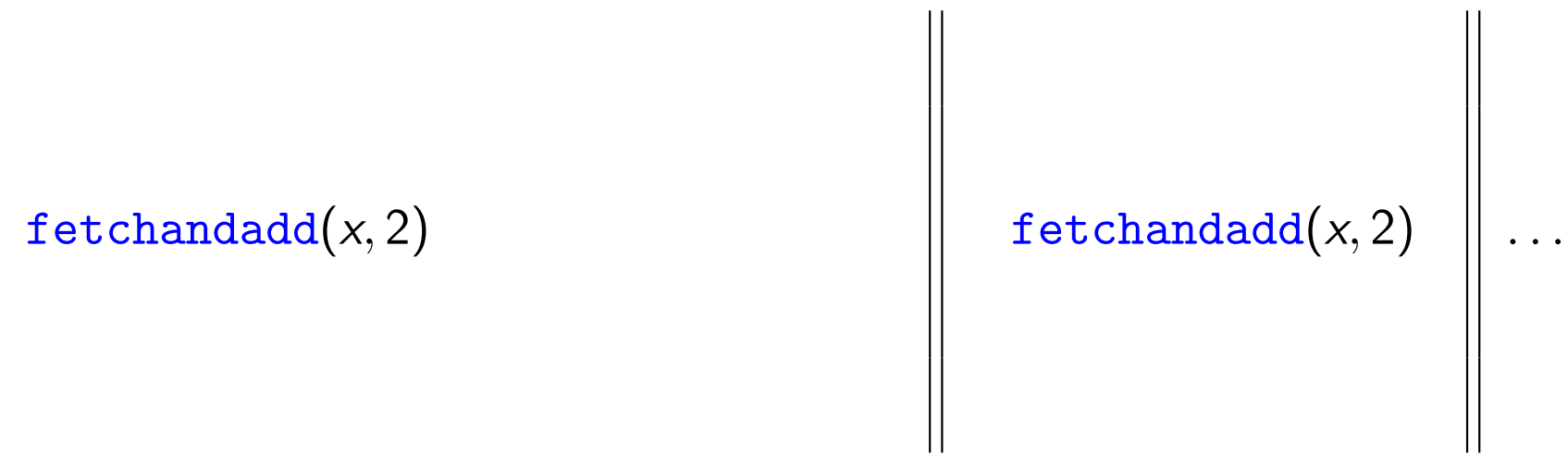
$\{x \mapsto 0\}$



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$\left\{x \mapsto 0 * \gamma \hookrightarrow_\bullet 0 * \gamma \overset{1}{\hookrightarrow}_\circ 0\right\}$

$\texttt{fetchandadd}(x, 2)$ || $\texttt{fetchandadd}(x, 2)$ || $\ldots$

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allocate $\boxed{\exists n.\, x \mapsto n * \gamma \hookrightarrow_\bullet n}$

$\texttt{fetchandadd}(x, 2)$ || $\texttt{fetchandadd}(x, 2)$ || ...

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$\left\{\gamma \overset{1/k}{\hookrightarrow}_\circ 2\right\}$ $\quad\|\quad$ $\left\{\gamma \overset{1/k}{\hookrightarrow}_\circ 2\right\}$ $\quad\|$

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$\texttt{fetchandadd}(x, 2)$
$\left\{\gamma \overset{1/k}{\hookrightarrow}_\circ 2 * x \mapsto (2+n) * \gamma_1 \hookrightarrow_\bullet (2+n)\right\}$
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$\parallel$

$\left\{\gamma \overset{1/k}{\hookrightarrow}_\circ 0\right\}$
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$\Big\|$

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$\texttt{fetchandadd}(x, 2)$

$\{\ldots\}$

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The Iris story, Part 2:
Modeling ghost state via "PCMs"

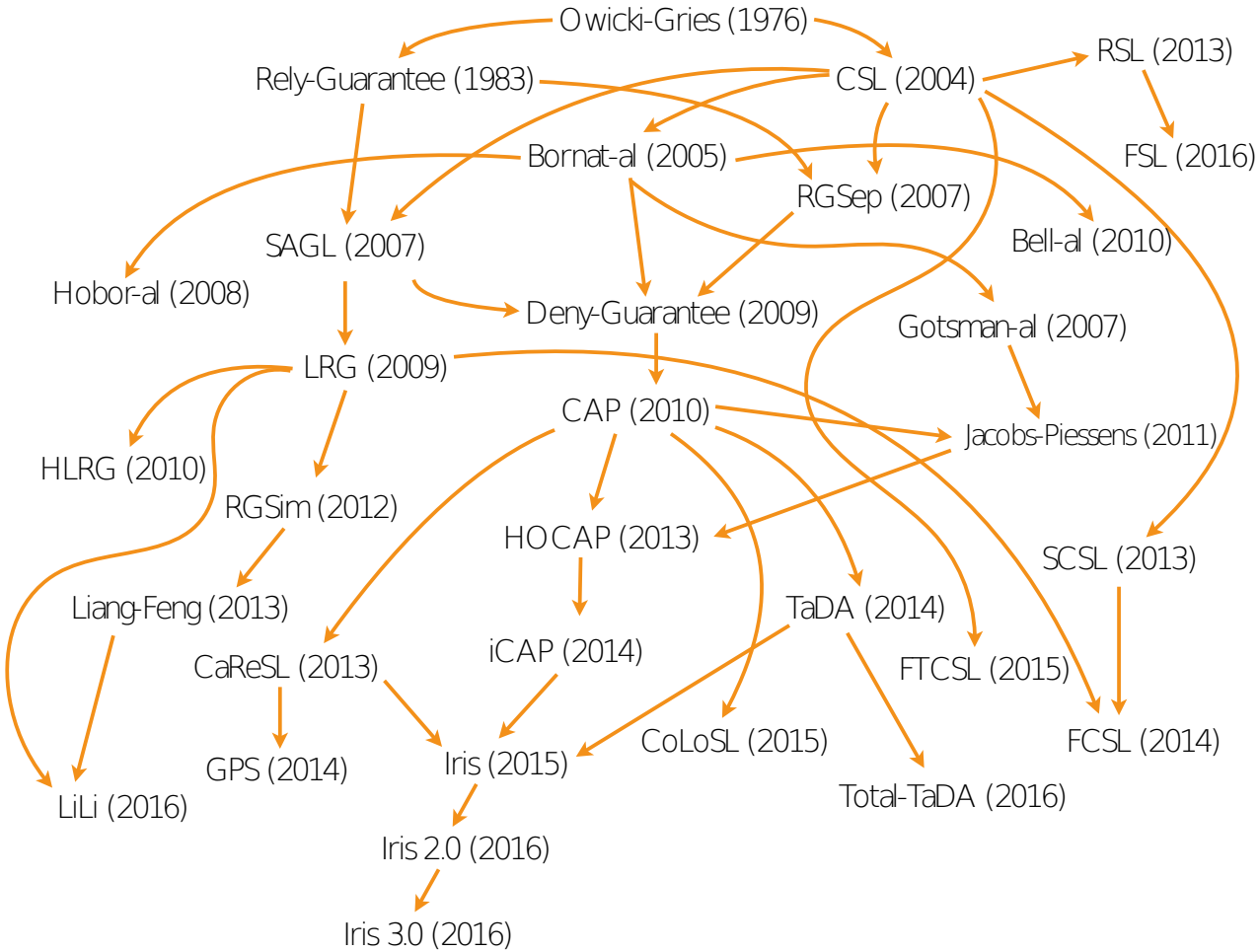
Mechanisms for concurrent reasoning

We have seen so far:

- Invariants $\boxed{R}^{\mathcal{N}}$
- Ghost variables $\gamma \hookrightarrow_\bullet n$ and $\gamma \hookrightarrow_\circ n$
- Fractional ghost variables $\gamma \hookrightarrow_\bullet n$ and $\gamma \overset{\pi}{\hookrightarrow}_\circ n$

Where do these mechanisms come from?

There are many CSLs with more powerful mechanisms...



Picture by Ilya Sergey

. . . and very complicated **primitive** rules

$$\frac{\begin{array}{c}\Gamma,\Delta \mid \Phi \vdash \mathsf{stable}(\mathsf{P}) \qquad \Gamma,\Delta \mid \Phi \vdash \forall y.\ \mathsf{stable}(\mathsf{Q}(y)) \\ \Gamma,\Delta \mid \Phi \vdash n \in C \qquad \Gamma,\Delta \mid \Phi \vdash \forall x \in X.\ (x, f(x)) \in \overline{T(A)} \vee f(x) = x \\ \Gamma \mid \Phi \vdash \forall x \in X.\ (\Delta).\langle \mathsf{P} * \circledast_{\alpha\in A}[\alpha]^{\mathsf{n}}_{g(\alpha)} * \triangleright I(x)\rangle\ c\ \langle \mathsf{Q}(x) * \triangleright I(f(x))\rangle^{C\setminus\{n\}}\end{array}}{\begin{array}{c}\Gamma \mid \Phi \vdash (\Delta).\ \langle \mathsf{P} * \circledast_{\alpha\in A}[\alpha]^{\mathsf{n}}_{g(\alpha)} * \mathsf{region}(X,T,I,n)\rangle \\ c \\ \langle \exists x.\ \mathsf{Q}(x) * \mathsf{region}(\{f(x)\},T,I,n)\rangle^{C}\end{array}}\ \textsc{Atomic}$$

$$\frac{\mathcal{C} \vdash \forall b \overset{\text{rely}}{\sqsupseteq}_\pi b_0.\ (\!|\pi[\![b]\!] * P|\!)\ i \Rightarrow_1 a\ (\!|x.\ \exists b' \overset{\text{guar}}{\sqsupseteq}_\pi b.\ \pi[\![b']\!] * Q|\!)}{\mathcal{C} \vdash \left\{\boxed{b_0}^n_\pi * \triangleright P\right\}\ i \Rightarrow a\ \left\{x.\ \exists b'.\ \boxed{b'}^n_\pi * Q\right\}}\ \textsc{UpdIsl}$$

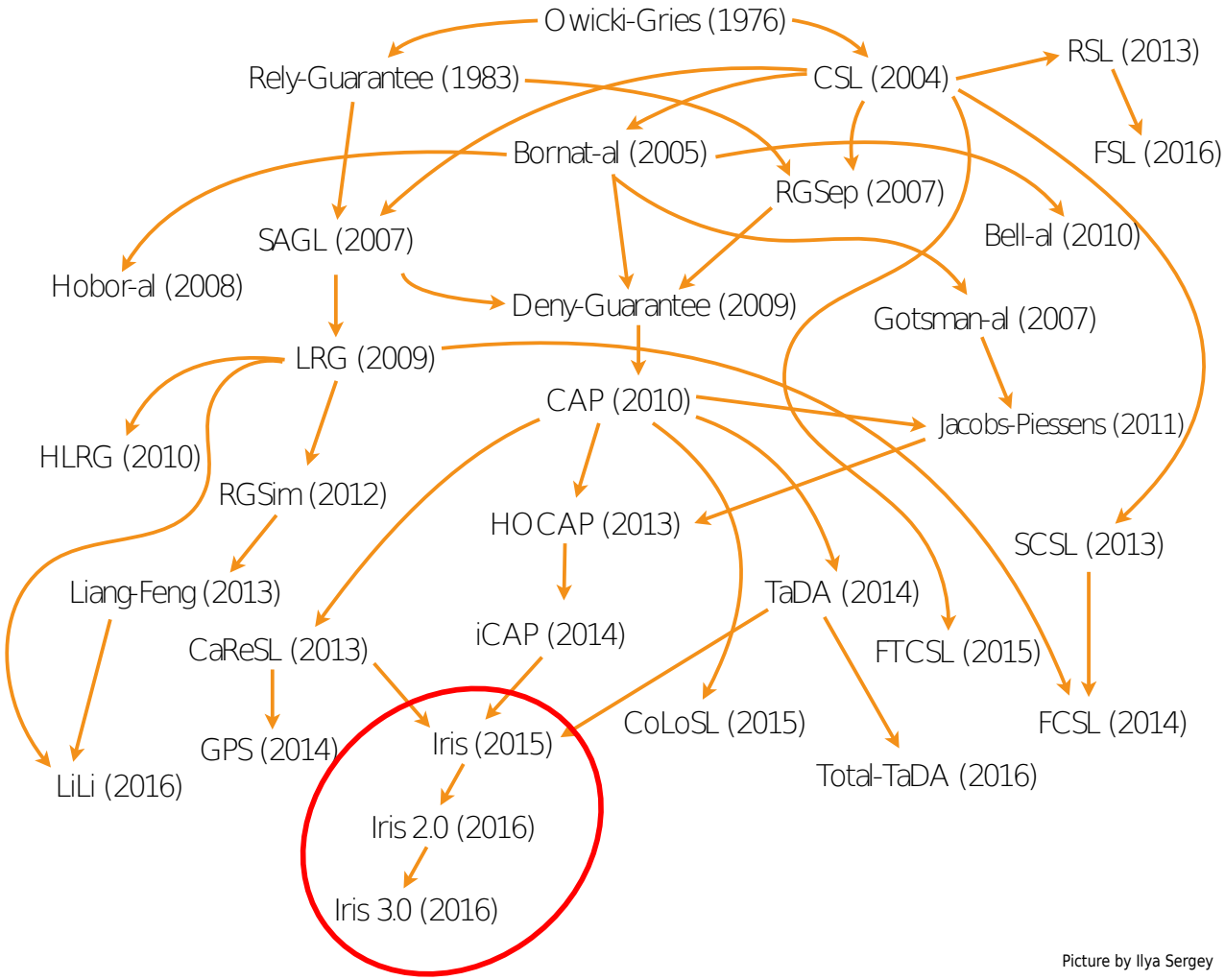
$$\frac{\begin{array}{c}\text{Use atomic rule} \\ a \notin \mathcal{A} \quad \forall x \in X.\ (x, f(x)) \in \mathcal{T}_{\mathbf{t}}(\mathrm{G})^* \\ \lambda; \mathcal{A} \vdash \mathbb{W}x \in X.\ \langle p_p \mid I(\mathbf{t}^\lambda_a(x)) * p(x) * [\mathrm{G}]_a\rangle\ \mathbb{C}\ \ \exists y \in Y.\ \langle q_p(x,y) \mid I(\mathbf{t}^\lambda_a(f(x))) * q(x,y)\rangle\end{array}}{\lambda+1; \mathcal{A} \vdash \mathbb{W}x \in X.\ \langle p_p \mid \mathbf{t}^\lambda_a(x) * p(x) * [\mathrm{G}]_a\rangle\ \mathbb{C}\ \ \exists y \in Y.\ \langle q_p(x,y) \mid \mathbf{t}^\lambda_a(f(x)) * q(x,y)\rangle}$$

$$\frac{\begin{array}{c}\Gamma \mid \Phi \vdash x \in X \qquad \Gamma \mid \Phi \vdash \forall \alpha \in \mathsf{Action}.\ \forall x \in \mathsf{SId} \times \mathsf{SId}.\ up(T(\alpha)(x)) \\ \Gamma \mid \Phi \vdash A \text{ and } B \text{ are finite} \qquad \Gamma \mid \Phi \vdash C \text{ is infinite} \\ \Gamma \mid \Phi \vdash \forall n \in C.\ \mathsf{P} * \circledast_{\alpha\in A}[\alpha]^n_1 \Rightarrow \triangleright I(n)(x) \\ \Gamma \mid \Phi \vdash \forall n \in C.\ \forall s.\ \mathsf{stable}(I(n)(s)) \qquad \Gamma \mid \Phi \vdash A \cap B = \emptyset\end{array}}{\Gamma \mid \Phi \vdash \mathsf{P} \sqsubseteq^C \exists n \in C.\ \mathsf{region}(X,T,I(n),n) * \circledast_{\alpha\in B}[\alpha]^n_1}\ \textsc{VAlloc}$$

$$\frac{\begin{array}{c}\text{Update region rule} \\ \lambda; \mathcal{A} \vdash \mathbb{W}x \in X.\ \left\langle p_p \,\middle|\, I(\mathbf{t}^\lambda_a(x)) * p(x)\right\rangle\ \mathbb{C}\ \ \exists y \in Y.\ \left\langle q_p(x,y) \,\middle|\, \begin{array}{c} I(\mathbf{t}^\lambda_a(Q(x))) * q_1(x,y) \\ \vee\ I(\mathbf{t}^\lambda_a(x)) * q_2(x,y)\end{array}\right\rangle\end{array}}{\begin{array}{l}\mathbb{W}x \in X.\ \langle p_p \mid \mathbf{t}^\lambda_a(x) * p(x) * a \Mapsto \blacklozenge\rangle \\ \mathbb{C} \\ \exists y \in Y.\ \left\langle q_p(x,y) \,\middle|\, \begin{array}{c}\exists z \in Q(x).\ \mathbf{t}^\lambda_a(z) * q_1(x,y) * a \Mapsto (x,z) \\ \vee\ \mathbf{t}^\lambda_a(x) * q_2(x,y) * a \Mapsto \blacklozenge\end{array}\right\rangle\end{array}}$$

$$\lambda{+}1; a : x{\in}X \rightsquigarrow Q(x), \mathcal{A} \vdash$$

The Iris story



The Iris story: many of these mechanisms can be **encoded** using a simple mechanism of *resource ownership*

Generalizing ownership

All forms of ownership have common properties:

- Ownership of different threads can be composed
  For example:

$$\gamma \xhookrightarrow{\pi_1+\pi_2}_{\circ} (n_1 + n_2) \quad \Leftrightarrow \quad \gamma \xhookrightarrow{\pi_1}_{\circ} n_1 * \gamma \xhookrightarrow{\pi_2}_{\circ} n_2$$

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- Composition of ownership is associative and commutative
  Mirroring that parallel composition and separating conjunction is associative and commutative
- Combinations of ownership that do not make sense are ruled out
  For example:

$$\gamma \hookrightarrow_{\bullet} 5 * \gamma \xhookrightarrow{1/2}_{\circ} 3 * \gamma \xhookrightarrow{1/2}_{\circ} 4 \quad \Rightarrow \quad \text{False}$$

  (because $5 \neq 3 + 4$)

Resource algebras (RAs): A generalization of PCMs

Resource algebra (RA) with carrier M:

- Composition $(\cdot) : M \to M \to M$
- Validity predicate $\mathcal{V} \subseteq M$

Satisfying:

$$a \cdot b = b \cdot a \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad (a \cdot b) \in \mathcal{V} \Rightarrow a \in \mathcal{V}$$

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Iris has ghost variables $\boxed{a : M}^{\gamma}$ for each resource algebra M

$$a \in \mathcal{V} \Rrightarrow \exists \gamma.\, \boxed{a}^{\gamma} \qquad \boxed{a}^{\gamma} * \boxed{b}^{\gamma} \Leftrightarrow \boxed{a \cdot b}^{\gamma} \qquad \boxed{a}^{\gamma} \Rightarrow \mathcal{V}(a)$$

$$\frac{\forall a_{\mathrm{f}}.\, a \cdot a_{\mathrm{f}} \in \mathcal{V} \Rightarrow b \cdot a_{\mathrm{f}} \in \mathcal{V}}{\boxed{a}^{\gamma} \Rrightarrow \boxed{b}^{\gamma}}$$

Ghost variables revisited

Resource algebra for ghost variables:

$$M \triangleq \bullet\, n \mid \circ\, n \mid \bot \mid \bullet\!\circ\, n$$

$$\mathcal{V} \triangleq \{a \neq \bot \mid a \in M\}$$

$$\bullet\, n \cdot \circ\, n' = \circ\, n' \cdot \bullet\, n \triangleq \begin{cases} \bullet\!\circ\, n & \text{if } n = n' \\ \bot & \text{otherwise} \end{cases}$$

$$\text{other combinations} \triangleq \bot$$

And define:

$$\gamma \hookrightarrow_\bullet n \triangleq \boxed{\bullet\, n}^{\gamma} \qquad \gamma \hookrightarrow_\circ n \triangleq \boxed{\circ\, n}^{\gamma}$$

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The ghost variable rules follow directly from the general rules:

$$\text{True} \Rrightarrow \exists \gamma.\, \gamma \hookrightarrow_\bullet n * \gamma \hookrightarrow_\circ n$$

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$$\gamma \hookrightarrow_\bullet n * \gamma \hookrightarrow_\circ m \Rightarrow (\bullet\, n \cdot \circ\, m) \in \mathcal{V} \Rightarrow n = m$$

Updating resources

Resources can be *updated* using *frame-preserving updates:*

$$\frac{\forall a_{\mathrm{f}}.\, a \cdot a_{\mathrm{f}} \in \mathcal{V} \Rightarrow b \cdot a_{\mathrm{f}} \in \mathcal{V}}{\boxed{a}^{\gamma} \Rrightarrow \boxed{b}^{\gamma}}$$

Key idea: a resource can be updated if the update does not invalidate the resources of concurrently-running threads

Thread 1		Thread 2		...		Thread n	
a_1	$\cdot$	a_2	$\cdot$	$\ldots$	$\cdot$	a_n	$\in \mathcal{V}$
$\rightsquigarrow$							
b_1	$\cdot$	a_2	$\cdot$	$\ldots$	$\cdot$	a_n	$\in \mathcal{V}$

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$\rightsquigarrow$							
b_1	$\cdot$	a_2	$\cdot$	$\ldots$	$\cdot$	a_n	$\in \mathcal{V}$

The rule $\gamma \hookrightarrow_\bullet n * \gamma \hookrightarrow_\circ m \Rrightarrow \gamma \hookrightarrow_\bullet n' * \gamma \hookrightarrow_\circ n'$ follows directly

Generalizing to a library of RA combinators

Iris comes with a library of useful RA combinators

- $\textsc{Auth}(M)$: Generalizes the $\bullet$, $\circ$, $\bullet\circ$ construction over an arbitrary RA M – we call it the "authoritative" RA.
- $\textsc{Frac}$: The RA for fractions in $(0, 1]$ with addition.
- $\textsc{Excl}(X)$: The "exclusive" RA, whose valid elements are the elements of X, and where composition is always undefined.
- The expected RA liftings of products, sums, etc.

Using these combinators, we can easily construct the necessary models of many desired forms of ghost state:

- Ghost variables from this talk: $\textsc{Auth}\,(\textsc{Excl}\,\textsc{Nat})$
- Fractional ghost variables: $\textsc{Auth}\,(\textsc{Frac} \times \textsc{Nat}_+)$

Many things I haven't covered

Modal basis of Iris: $\Box$, $\triangleright$, $\Rightarrow\!\!\!\!\mid$

- Persistent modality $\Box P$: Says P holds forever, i.e., only relying on duplicable resources, such as invariants
- Later modality $\triangleright P$: Says P holds one step-index later (lower); needed to model impredicative invariants
- Update modality $\Rightarrow\!\!\!\!\mid P$: Says P holds after some frame-preserving update to ghost state

Higher-order ghost state, e.g., named propositions $\gamma \mapsto P$

- $\gamma \mapsto P * \gamma \mapsto Q \Rightarrow \triangleright(P = Q)$
- Sounds arcane, but turns out to be surprisingly useful!
- Achieved by equipping RAs with a step-indexing structure

Encoding of Iris program logic (including invariants)
within the modal base logic (with higher-order ghost state)

Conclusion

The Iris methodology for concurrent reasoning:

- Divide up logical ownership of shared physical state using appropriately chosen **ghost state predicates and axioms**
- Tie ghost state assertions to physical state using **invariants**
- Build model of ghost state predicates by choosing an appropriate (step-indexed) **"PCM"**
- Verify ghost state axioms as instances of a few basic laws like **frame-preserving update**

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Yes, but is it "canonical"?

I'll let you be the judge!